Zastosowanie mieszaniny dwóch rozkładów gamma do audytu finansowego
Application of two gamma distributions mixture to financial auditing

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Model of accounting observations
Basic symbols

- U-population of size N, s-sample of size $n \leq N$;
- vector of book values (auxiliary variable): $\boldsymbol{x} = [x_1 ... x_N]$, $\boldsymbol{x} \in R_+^N$ is outcome of $\boldsymbol{X}^T = [X_1 ... X_N]$;
- true (without errors) accounting amounts: $\mathbf{y} = [y_1 ... y_N]$, $\boldsymbol{y} \in R_+^N$ is outcome of $\boldsymbol{Y}^T = [Y_1 ... Y_N]$;
- accounting amounts contaminated by errors: $\mathbf{w} = [w_1 ... w_N]$, $\boldsymbol{w} \in R_+^N$ is outcome of $\boldsymbol{W}^T = [W_1 ... W_N]$;
- $\mathbf{Z}^T = [Z_1 ... Z_N]$, $Z_i = 0$ $(Z_i = 1) \Leftrightarrow X_i = Y_i$ $(X_i = W_i)$, $i \in U$;

Model of accounting observations
Assumptions

- Rows: $[X_i\ Y_i\ W_i\ Z_i]$ of matrix [**X Y W Z**] are independent and identically distributed as random vector $[X\ Y\ W\ Z]$;

$$X = (1 - Z)Y + ZW \qquad \text{or} \qquad X = Y + ZR \tag{1}$$

$R = W - Y$ is the auditing error, we assume that $R \geq 0$;

- The probability distribution of the matrix [**X Y W Z**] is the population model.
- Distribution of X is the following mixture of distributions:

$$F(x|\theta) = (1 - p)F_0(x|\theta_0) + pF_1(x|\theta_1), \tag{2}$$

$P(Z = 1) = p$, $P(Z = 0) = 1 - p$;
$F_0(x|\theta_0) = F(x|Z = 0) = F_0(y|\theta_0)$,
$F_1(x|\theta_1) = F(x|Z = 1) = F_1(w|\theta_1)$,
$\theta = \theta_0 \cup \theta_1$, $\theta \in \Theta = \Theta_0 \cup \Theta_1$,

$$f(x|\theta) = (1 - p)f_0(x|\theta_0) + pf_1(x|\theta_1). \tag{3}$$

Model of accounting observations
Purpose of inference

- Expected mean accounting error:

  $\tau = E(\bar{X} - \bar{Y}) = p(E(W|\theta_1) - E(Y|\theta_0))$

  or expected total account. error:

  $N\tau = E(\sum_{i \in U} X_i - \sum_{i \in U} Y_i)$;

- Hypotheses: $H_0: \quad \tau = \tau_0, \qquad H_1: \quad \tau = \tau_1 > \tau_0$

  τ_0, (τ_1): admissible, (un-admissible) exp. account. error.

- Let α be significance level (risk of incorrect rejection of H_0),

  $(1 - \beta)$-probability of II kind error (risk of incorrect acceptance of H_0) where β-power of the test.

Model of accounting observations
Data

- Before auditing process the following data are observed:

$$\boldsymbol{X} = (X_i : i \in U) = (\boldsymbol{X}_s, \boldsymbol{X}_{U-s})$$

where

$$\boldsymbol{X}_s = (X_i : i \in s), \quad \boldsymbol{X}_{U-s} = (X_i : i \in U - s)$$

- After the auditing process the following data are observed:

$$\mathcal{D} = (\mathcal{D}_s, \boldsymbol{X}_{U-s}), \quad \mathcal{D}_s = ((X_i, Z_i) : i \in s) = (\boldsymbol{Y}_{s_0}, \boldsymbol{W}_{s_1}).$$

$\boldsymbol{d}$, $\boldsymbol{d}_s$, $\boldsymbol{x}$, $\boldsymbol{x}_s$, $\boldsymbol{x}_{U-s}$, $\boldsymbol{y}_{s_0}$ and $\boldsymbol{w}_{s_1}$ are outcomes of $\mathcal{D}$, $\mathcal{D}_s$, $\boldsymbol{X}$, $\boldsymbol{X}_s$, $\boldsymbol{X}_{U-s}$, $\boldsymbol{Y}_{s_0}$ and $\boldsymbol{W}_{s_1}$, respectively.

- Let: Let $Y \sim G(a, c)$ and $R \sim G(b, c)$ be independent, then variable $W = Y + R \sim G(a + b, c)$;
- the mixture:

$$f(x|a, b, c, p) = pf_1(x|a, b, c) + (1 - p)f_0(x|a, c) \qquad (4)$$

where

$$f_1(x|a, b, c) = \frac{c^{a+b}}{\Gamma(a+b)} x^{a+b-1} e^{-cx}, \quad f_0(x|a, c) = \frac{c^a}{\Gamma(a)} x^{a-1} e^{-cx},$$

$$\tau = p(E(X|a, b, c, p) - E(Y|a, c)) = \frac{pb}{c}. \qquad (5)$$

Moment method of estimation

The sample s is not selected, $s = \varnothing$

- The solution $\{p_U(x), a_U(x), b_U(x), c_U(x)\}$ of the equation system, Wywiał(2016, 2018):

$$E(X^e) = m_e(x), \quad e = 1, 2, 3, 4, \quad N > 4,$$

$m_e(x) = \frac{1}{N} \sum_{i \in U} x_i^e$, is the estimator of $\{p, a, b, c\}$.

- Test statistic:

$$\hat{G}_1 = \frac{\hat{\tau}_1 - \tau_0}{\sqrt{Q_U(\mathcal{D})}}, \qquad \hat{\tau}_1 = \frac{p_U b_U}{c_U},$$

$Q_U(\mathcal{D})$ - e.g. bootstrap type estimator.

- p-value could be evaluated based on limit distribution of $\hat{G}_1$ or Monte-Carlo procedures.

Moment method of estimation

The sample s is not empty, $s = s_0 \cup s_1$, $s_0 \neq \varnothing$, $s_1 \neq \varnothing$

- Test statistic, Wywiał(2018):

$$\hat{G}_2 = \frac{\hat{\tau}_2 - \tau_0}{\sqrt{\frac{V_{U-s}(X)}{N-n} + \frac{V_{s_0}(Y)}{n_0}}}, \qquad \hat{\tau}_2 = \bar{X}_{U-s} - \bar{Y}_{s_0}. \tag{6}$$

- $V_z(T)$ is variance of variable T observations in set $z \subseteq U$, $\{P_U, A_{s_0}, B_s, C_{s_0}\}$ are estimators of $\{p, a, b, c\}$ where:

$$\begin{cases} P_U = \frac{\bar{X}_{U-s} - \bar{Y}_{s_0}}{\bar{R}_{s_1}}, & A_{s_0} = \frac{\bar{Y}_{s_0}^2}{V_{s_0}(Y)}, \\ B_s = \frac{\bar{Y}_{s_0}\bar{R}_{s_1}}{V_{s_0}(Y)}, & C_{s_0} = \frac{\bar{Y}_{s_0}}{V_{s_0}(Y)} \end{cases} \tag{7}$$

  provided denominators of the above ratios are positive.

- p-value could be evaluated based on limit distribution of $\hat{G}_2$ or Monte-Carlo methods.
- Cases: ($s_0 \neq \varnothing$, $s_1 = \varnothing$) and ($s_0 = \varnothing$, $s_1 \neq \varnothing$) are considered by Wywiał (2016)

- Log-likelihood function:

$$l(\boldsymbol{d}|\theta) = ln(L(\boldsymbol{d}|\theta)) = kln(p) + (n-k)ln(1-p) + \\ + \sum_{i \in s_1} ln(f_1(x_i|\theta_1)) + \sum_{i \in s_0} ln(f_0(x_i|\theta_0)) + \sum_{i \in U-s} ln(f(x_i|\theta)).$$

- Log-likelihood function in the case of gamma-mixture distribution:

$$l(\boldsymbol{d}, a, b, c, p) = k\ ln(p) + (n-k)ln(1-p) + Na\ ln(c) + kb\ ln(c) + \\ -k\ ln(\Gamma(a+b)) - (n-k)ln(\Gamma(a)) + (a-1)\sum_{j \in U} ln(x_j) + b\sum_{j \in s_1} ln(x_j) + \\ - c\sum_{j \in U} x_j + \sum_{j \in U-s} ln\left(\frac{1-p}{\Gamma(a)} + \frac{p(cx_j)^b}{\Gamma(a+b)}\right). \quad (8)$$

Likelihood ratio test
Test statistic

- Likelihood ratio statistic:

$$\lambda = \frac{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}, \tau(\theta) = \tau_0} L(\mathcal{D}|\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} L(\mathcal{D}|\boldsymbol{\theta})}. \tag{9}$$

- Distribution of $-ln(\lambda)$ is approximated by chi-square distribution with 1 degree of freedom under the sufficiently large size of sample.

Monte Carlo test 1

Parameter of gamma distribution estimated by method of moments

- $f(x|a,b,c,p)$ is transformed by means $c = \frac{pb}{\tau}$ into:

$$f(x|a,b,\tau,p) = pf_1(x|a,b,\tau) + (1-p)f_0(x|a,\tau).$$

- Parameters a, b are replaced with estimators given by (7), see Dufour (2006).
- Data $\boldsymbol{d}^{(0,i)} = \left(\boldsymbol{y}_{s_0}^{(0,i)}, \boldsymbol{w}_{s_1}^{(0,i)}, \boldsymbol{x}_{U-s}^{(0,i)}, \right)$, $i = 1, ..., m$, are generated according to $f_0(y|a_{s_0}, \tau_0)$ with probability $(1-p)$ and $f_1(w|a_{s_0}, b_s, \tau_0)$ with prob. p;
- Data $\boldsymbol{d}^{(1,i)} = \left(\boldsymbol{y}_{s_0}^{(1,i)}, \boldsymbol{w}_{s_1}^{(1,i)}, \boldsymbol{x}_{U-s}^{(1,i)}, \right)$ are generated m-times according to $f_0(y|a_{s_0}, \tau_1)$ with probab. $(1-p)$ and $f_1(w|a_{s_0}, b_s, \tau_1)$ with prob. p;

Monte Carlo test 2

Simulated distribution of the test statistic

- Test statistic, Wywiał(2018):

$$\hat{g}_2^{(e,i)} = \frac{\tau^{(e,i)} - \tau_0}{\sqrt{\frac{v_{U-s}\left(\boldsymbol{x}^{(e,i)}\right)}{N-n} + \frac{v_{s_0}\left(\boldsymbol{y}^{(e,i)}\right)}{n_0}}}, \quad \tau^{(e,i)} = \bar{X}_{U-s}^{(e,i)} - \bar{Y}_{s_0}^{(e,i)}, \ e = 0, 1.$$

- Data $\{\hat{g}_2^{(e,i)}, i = 1, ..., m\}$ approximates the distrib. of $\hat{G}_2$ when hypothesis H_e is true, $e = 0, 1$;
- Let (see: Dufour and Khalaf (2001)):

$$\eta_e = \frac{m\omega_e}{m+1}, \quad \omega_e = \frac{1}{m}\sum_{i=1}^{m} I(\hat{g}_2^{(e,i)}), \quad I(\hat{g}_2^{(e,i)}) = \begin{cases} 1, & \text{if } \hat{g}_2^{(e,i)} \geq \hat{g}_2 \\ 0, & \text{if } \hat{g}_2 <^{(e,i)} \hat{g}_2 \end{cases}$$

ω_e is equal to the frequency of appearing inequalities $\hat{g}_2^{(e,i)} \geq \hat{g}_2$, $i = 1, ..., m$, $e = 0, 1$.

Monte Carlo test 3

Simulated distribution of the test statistic. Decisions

- *p*-value the power of the test is assessed by $\hat{\alpha} = \eta_0$, and $\hat{\beta} = \eta_1$, respectively.
- If $\hat{\alpha} \leq \alpha$, H_0 is rejected, α is the risk of incorrect rejection;
- If $\hat{\alpha} > \alpha$, H_0 is accepted, $(1 - \hat{\beta})$ is the risk of incorrect acceptance.

Monte Carlo test 4

Reparametrization of the ikelihood function

- After substituting c for $\frac{pb}{\tau}$ in $l(\boldsymbol{d}, a, b, c, p)$, (see expr. (8)):

$$l(\boldsymbol{d}, a, b, \tau, p) = k\ ln(p) + (n-k) ln(1-p) + Na\ ln(c) + kb\ ln(c) +$$
$$-k\ ln(\Gamma(a+b)) - (n-k) ln(\Gamma(a)) + (a-1) \sum_{j \in U} ln(x_j) + b \sum_{j \in s_1} ln(x_j) +$$
$$- \frac{pb}{\tau} \sum_{j \in U} x_j + \sum_{j \in U-s} ln\left(\varphi(a, b, \tau, p, x_j)\right) \quad (10)$$

where

$$\varphi(a, b, \tau, p, x_j) = \frac{1-p}{\Gamma(a)} + \frac{p^{b+1}(bx_j)^b}{\tau^b \Gamma(a+b)}.$$

- Now: $E(X) = \tau$.

Monte Carlo test 5

Simulated distribution of the likelihood ratio test

- The likelihood ratio test statistic, Wywiał(2018):

$$t = 2\left(l(\boldsymbol{d}, \hat{a}, \hat{b}, \hat{\tau}, \hat{p}) - l(\boldsymbol{d}, \tilde{a}, \tilde{b}, \tau_e, \tilde{p})\right)$$

$(\hat{a}, \hat{b}, \hat{\tau}, \hat{p})$ maximizes $l(\boldsymbol{d}, a, b, \tau, p)$ (see: (10)),
$(\tilde{a}, \tilde{b}, \tilde{p})$ maximizes $l(\boldsymbol{d}, a, b, \tau_0, p)$.

- $\boldsymbol{d}^{(e,i)}$, $e = 0, 1$, is generated according to $f(x|\tilde{a}, \tilde{b}, \tau_e, \tilde{p})$, $i = 1, ..., m$.
- Simulated distribution of test statistic:

$$t_i^{(e)} = 2\left(l(\boldsymbol{d}^{(e,i)}, \hat{a}^{(i)}, \hat{b}^{(i)}, \hat{\tau}^{(i)}, \hat{p}) - l(\boldsymbol{d}^{(e,i)}, \tilde{a}^{(i)}, \tilde{b}^{(i)}, \tau_e, \tilde{p}^{(i)})\right),$$

$(\hat{a}^{(i)}, \hat{b}^{(i)}, \hat{\tau}^{(i)}, \hat{p}^{(i)})$ maximizes $l(\boldsymbol{d}^{(e,i)}, a, b, \tau, p)$,
$(\tilde{a}^{(i)}, \tilde{b}^{(i)}, \tilde{p}^{(i)})$ maximizes $l(\boldsymbol{d}^{(e,i)}, a, b, \tau_e, p)$.

Monte Carlo test 6

Simulated distribution of the likelihood ratio test. Decisions

- t_α is the critical value of the test defined as the sample quantile of order $(1-\alpha)$ of $\{t_i^{(0)}, i = 1, ..., m\}$;
- $\hat{\beta}$, is the power evaluated as the frequency of appearing inequalities $t_i^{(1)} \geq t_\alpha$, $i = 1, ..., m$;
- if $t \geq t_\alpha$, H_0 is rejected, α is the risk of incorrect rejection;
- if $t < t_\alpha$, H_0 is accepted, $(1 - \hat{\beta})$ is the risk of incorrect acceptance.

Conclusions

- Model of accounting data is defined as mixture of two distributions.
- In particular, the mixture of two gamma distribution is considered.
- Hypothesis on the mean accounting error is tested by means of studentized estimator of the mean or likelihood ratio test.
- Monte-Carlo methods of approximation distributions of test statistics is preferable.
- Specification of the alternative distribution let us control the error of II kind (risk of incorrect acceptance).
- It is possible to test the hypothesis without auditing process of data provided the mixture model is true.
- Mixtures of other distributions can be considered.
- The considered problem can be generalized into model-design approach (see Wywiał(2016)).

Reference

- Cox D. R., Snell E. J. (1979). On sampling and the estimation of rare errors. *Biometrika*, 66, No. 1, pp. 125-132. Errata: *Biometrika*, 1982, 69, No. 2, p. 491.
- Dufour J. M. (2006). Monte Carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics. *Journal of Econometrics* vol. 133, pp. 443-477.
- Dufour J. M., Khalaf L. (2001). Monte Carlo test methods in econometrics. In: *Companion to Theoretical Econometrics*, ed. B. Baltagi. Oxford, U.K., pp. 494-519.
- Fienberg S. E., Nether J., Leitch R. A. (1977). Estimating the total overstatement error in accounting populations. *Journal of the American Statistical Association*, vol. 72, pp. 295-302.
- Frost P. A., Tamura H. (1986). Accuracy of auxiliary information interval estimation in statistical auditing. *Journal of Accounting Research* vol. 24, pp. 57-75.

Reference

- Hall P. (1992). *The Bootstrap and Edgewrth Expansion.* Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest.
- Hope C.A. (1968). A simplified Monte-Carlo significance test procedure. *Journal of the Royal Statistical Society* B30, no. 3, 582-598.
- MacKinnon J. (2007): Bootstrap hypothesis testing. Queen's Economics Department, *Working Paper* no. 1127.
- Marazzi A., Tillé Y. (2016). Using past experience to optimize audit sampling design. *Review of Quantive Finance Accounting* p. 1-28, doi:10.1007/s11156-016-0596-7.
- McLachlan G., Peel D. (2000). *Finite Mixture Models*. John Wiley & Sons, Inc. New York Chichester Weinheim Brisbane Singapore Toronto.

Reference

- Rao C. R. (1973). *Linear Statistical Inference and Its Applications*. John Wiley & Sons, New-York - London - Sydney - Toronto.
- Silvey S. D. (1959). The Lagrangian multiplier test. *The Annals of Mathematical Statistics* vol. 30, no. 2, pp. 389-407.
- Statistical models and analysis in auditing. Panel on Nonstandard Mixtures of Distributions. (1989). *Statistical Science*, vol.4, nr 1, 2-33.
- Wywiał J. L. (2016). *Contributions to Testing Statistical Hypotheses in Auditing*. PWN, Warsaw.
- Wywiał, J. L. (2018). ***Application of two gamma distributions mixture to financial auditing. Sankhya B***, vol. 80, issue 1, 1-18.

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